SPATIAL-SCALING-COMPATIBLE MORPHOLOGICAL GRANULOMETRIES ON LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

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ABSTRACT

Ganulometries are defined on function classes over topological vector spaces. The usual Euclidean property for scaling compatibility, set scaling in the binary case and graph scaling in the gray-scale case, is changed so that it is with respect to spatial (domain) scaling for function spaces. As in the binary case, scaling compatible granulometries possess representations as double suprema over scaled generating elements. Without further constraint on the generating elements, the double supremum involves, for each generating element, all scalings exceeding the parameter of the particular granulometric operator. The salient theorem of the present paper concerns necessary and sufficient conditions under which there is a reduction of the double-supremum representation to a single supremum over singularly scaled generating functions. Specifically, and in the context of locally convex topological vector spaces, there is a determination of when a domain-scaled function $t*f$ is f-open for all $t \geq 1$. Key roles are played by both topology and local convexity.

1. INTRODUCTION

Morphological granulometries were introduced by Matheron [1] to model the sieving of a random binary image according to the size and shape of grains within the image. Intuitively, as the mesh size of the sieve is increased, more of the image grains will fall through the sieve and the residual area of the filtered (sieved) image will decrease monotonically. These residual areas form a *size distribution* that is indicative of image structure. Upon normalization, this size distribution becomes an increasing function from 0 to 1 and is a probability distribution function. Both it, and its derivative, which is a probability density, are called the *granulometric size distribution* of the image. Moments of this size distribution serve as image features. For instance, they can be employed for texture-based segmentation (Dougherty et al [2]). The basic theory of granulometric size distributions is discussed by Serra [3] and Dougherty and Giardina [4, 5].

Euclidean granulometries merit special attention in Matheron's original theory. These satisfy a certain property that makes them compatible (in a certain sense) with Euclidean scaling. A fundamental proposition of Matheron [1] is that the Euclidean granulometries possess a representation in terms of morphological openings. As it stands, this representation requires a

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double union that makes it impractical for application. A key question concerns conditions under which this double union reduces to a tractable single union, for when it does it provides a practical paradigm for the construction of size distributions. A fundamental theorem of Matheron [1] gives necessary and sufficient conditions for this reduction.

As noted by Serra [6], the algebraic theory of granulometries extends at once to complete lattices, (and therefore to gray-scale images). Direct extension of the Euclidean theory to the gray scale is given by Dougherty [7]. Here we use the terminology "direct extension" in a specific manner. The theory of [7] employs a Euclidean scaling condition that is induced from the umbra formulation of gray-scale morphology and takes place relative to the image's graph. Consequently, whereas a binary Euclidean granulometry must be compatible with scaling, or magnification, of an n-dimensional binary image within Euclidean n-space, a *gray-scale Euclidean granulometry* must be compatible with scaling, or magnification, of the image graph within Euclidean (n + 1)-dimensional space. From the geometrical perspective that is relevent to the present paper, there must be scaling compatibility in both the domain and range. As in the Matheron theory, there is a representation; however, here it involves a double supremum. Application requires reduction of this double supremum to a single supremum and nonrestrictive suffcient conditions for this reduction are given in terms of umbra convexity.

But there is another way to consider scaling compatibility: the granulometry must be compatible with scaling, or magnification, only in the domain of the image. Whereas the basic "shape" of an image is unchanged under graph scaling, it is substantially changed under domain scaling. We will define a new type of gray-scale granulometry for which sizing is compatible with scaling only in the domain. Again we achieve a Matheron-type granulometric representation, and like in the Euclidean approach of [7] this is a double supremum. But the key result of the present paper is a significant generalization of Matheron's reduction theorem to provide necessary and sufficient conditions for reduction of the double supremum representation. Moreover, since basic to our result is an appeal to the Krein-Milman theorem, we see that the proper mathematical setting for the analysis of spatial-scaling-compatible granulometries is a locally convex topological vector space.

From a geometric and application perspective, the main theorem presented here is a limiting theorem, because it shows that if we desire granulometric compatibility with domain magnification, then we must pay a price: put simply, and with details to follow, gray-scale granulometries compatible with spatial scaling possess single-sumpremum representations only if they are generated by flat structuring elements. This constraint greatly restricts the kind of granulometric information that can be extracted if we insist upon sizing compatibility with domain magnification instead of graph magnification.

The present paper is a shortened version of a more complete analysis presented in [8], which provides further extensions of the granulometric concept, along with corresponding reduction conditions. There are differences, however. Herein we concentrate on the most basic form of the reduction theorem. Furthermore, we present an alternative derivation of the basic theorem, one that pays particular attention to geometry. Lastly, whereas in [8] the general development stays in the Euclidean setting, with concluding commentary pointing out the manner in which it is adapted to locally convex topological vector spaces, here we adhere closely to the topological-vector-space structure and pay careful attention to the manners in which topology and local convexity determine the main results.

2. THE CLASSICAL MATHERON THEORY

A one-parameter family of operators Ψ_t: **P** --> **P**, where t > 0 and **P** is the power set in R^n, is called a granulometry if Ψ_t is increasing, Ψ_t is antiextensive, and $\Psi_t \Psi_s = \Psi_s \Psi_t = \Psi_{max\{t,s\}}$. For any t > 0, the operator Ψ_t is an algebraic opening. The granulometry $\{\Psi_t\}$ is called a *τ-granulometry*, or T*-granulometry*, if each Ψ_t is translation invariant as an operator on **P**. To be consistent with [8] we employ the "T" terminology. $\{\Psi_t\}$ is called a *Euclidean granulometry* if it is a T-granulometry for which $\Psi_t(S) = t\,\Psi_1(S/t)$ for all t > 0.

If $\{\Psi_t\}$ is a T-granulometry, then it is a Euclidean granulometry if and only if $Inv[\Psi_t] = t \cdot Inv[\Psi_1]$, where $Inv[\Psi_t]$ is the invariant class of the algebraic opening Ψ_t. A collection of sets **G** is called a *generator* for a Euclidean granulometry if $Inv[\Psi_1]$ is the class closed under arbitrary union, translation, and scalar multiplication by t ≥ 1 generated by **G**. Letting O(S, B) denote the elementary morphological opening of S by B, the basic Matheron representation concerning Euclidean granulometries states that a family $\{\Psi_t\}$, t > 0, of operators on **P** is a Euclidean granulometry if and only if there exists a class **G** such that

$$\Psi_t(S) = \cup \{\cup \{O(S, rB): B \in \mathbf{G}\}: r \geq t\} \qquad (1)$$

G is a generator of $\{\Psi_t\}$.

Because the representation is necessary and sufficient, it provides a paradigm for constructing Euclidean granulometries. However, as it stands, evaluation of Ψ_t via the representation requires an infinite union over r ≥ t. This outer union is redundant if and only if O(S, rB) < O(S, tB) for all r ≥ t and B ∈ **G**, and this is equivalent to tB being B-open for all t ≥ 1. Matheron [1] has shown that, for compact sets, tB is B-open for all t ≥ 1 if and only if B is convex. This proposition is inherently topological becasue the compactness assumption cannot be dropped. Given this proposition (and given **G** is composed of compact sets), the representation (1) reduces to the single union

$$\Psi_t(S) = \cup \{O(S, tB): B \in \mathbf{G}\} \qquad (2)$$

if and only if all elements of **G** are convex.

3. UMBRA-INDUCED GRANULOMETRIES FOR GRAY-LEVEL IMAGES

As recognized by Serra [6], the definition of a binary granulometry applies at once for a family of operators $\{\Psi_t\}$ on a complete lattice. Thus, letting ≤ denote function ordering, the three basic granulometric axioms (antiextensivity, increasing, and $\Psi_t \Psi_r = \Psi_r \Psi_t = \Psi_{max\{t,r\}}$) apply at once to the gray scale, where here Ψ_t: Fun(V) --> Fun(V), the class of functions on the underlying Euclidean space V. Equivalently, a gray-scale granulometry is a one-parameter family $\{\Psi_t\}$ of gray-scale openings on Fun(V) such that $\Psi_t \leq \Psi_r$ if r ≥ t. $\{\Psi_t\}$ is called a T-granulometry if every Ψ_t is translation invariant in the usual gray-scale sense, namely, $\Psi_t(f_x + y) = \Psi_t(f)_x + y$, where $f_x + y$ is defined by $(f_x + y)(z) = f(z - x) + y$.

Our main concern is with the Euclidean property. Assuming $\{\Psi_t\}$ is a T-granulometry defined on R^n, Dougherty [7] has extended the Euclidean notion in a manner compatible with the umbra transform. For $t > 0$ and function f, define $t*f$ by $(t*f)(x) = tf(x/t)$. The *scalar multiplication* is based on scaling the graph of f as a subset of R^{n+1} and is umbra capatible in the sense that $U[t*f] = tU[f]$, where U[f] is the umbra of f. Given this definition of $t*f$, in [7] a *Euclidean gray-scale granulometry* is a T-granulometry for which

$$\Psi_t(f) = t*\Psi_1[t^{-1}*f] \quad (3)$$

for $t > 0$. From this definition, it is immediate that the invariant classes of a Euclidean granulometry $\{\Psi_t\}$ are determined by Inv[Ψ_1], just as in the binary setting. Moreover, it is shown that $\{\Psi_t\}$, $t > 0$, is a Euclidean granulometry if and only if there exists a class **G**, called the **generator** of $\{\Psi_t\}$, such that

$$\Psi_t(f) = \vee \{\vee \{O(f, r*g): g \in \mathbf{G}\}: r \geq t\} \quad (4)$$

where the openings in the representation are gray-scale openings. Inv[Ψ_1] is the closure of **G** under translations, suprema, and products $t*f$, $t \geq 1$. As in the binary case, elimination of the outer supremum relating to $r \geq t$ is crucial for application. This occurs if, for any $g \in \mathbf{G}$, $r*g$ is $t*g$-open for all $r \geq t$. If the graphs of the generator elements are concave-down, which makes their umbrae convex in R^{n+1}, then $r*g$ is $t*g$-open for $r \geq t$.

4. GRANULOMETRIES ON TOPOLOGICAL VECTOR SPACES

The present paper concerns a different gray-scale approach to the Euclidean property than the one taken in [7]; rather than define scaling in the domain space and range space relative to spatial scaling of the graph, and thereby have an umbra-based definition, herein we consider the problem of scaling only spatially. Regarding the domain space, four aspects are of concern: linear translation, scaling, topology, and convexity. We are also concerned with topology in the range space. While we could proceed by considering the domain space to be finite-dimensional Euclidean space and the range to be the extended real line, this would not do full justice to the full power of our main result. The fundamental point is that, relative to the kind of granulometric theory we now introduce, an appropriate setting is a topological vector space. Such a space carries the necessary mathematical structure for development of both translation-invariant granulometries and compatibility with spatial magnification. In addition, the main result of the present paper rquires the topological vector space to be locally convex. Those not wishing to concern themselves with the abstract theory of topological vector spaces can, whenever we mention such a space, simply substitute Euclidean space.

A vector space V endowed with a topology is a *topological vector space* if every point of V is a closed set and the vector space operations are continuous with respect to the topology. V is called a *locally convex topological vector space (lctvs)* if there exists a local base at 0 whose members are all convex. As we have defined it, the topology of a topological vector space is Hausdorff. Continuity of the vector space operations means that the mappings (x, y) --> x + y and (t, x) --> tx are continuous. In fact, for any point $a \in V$ and scalar t, the mappings x --> a + x and x --> tx are homeomorphisms from V --> V. Hence, the topology is translation invariant, so the topology is fully determined by any local base at 0. Herein we only consider real topological vector spaces. (See Rudin [9] for the basic theory of topological vector spaces.)

Given the structure of a lctvs V, gray-scale morphology is at once defined on Fun(V) by means of the usual gray-scale definitions. Indeed, the definition of a gray-scale T-granulometry carries over at once. Yet one need not necessarily extend the Euclidean condition in accordance with umbra (graph) scaling; a second way to proceed is to consider compatibility with scaling in the domain only. Here there is spatial scaling (corresponding to spatial magnification) but not gray-level scaling. The appropriate definition of the domain scaling $t*f$ is given by $(t*f)(x) = f(x/t)$. Henceforth we employ only this definition of scaling (so there should be no confusion with the notation $*$). A T-granulometry $\{\Psi_t\}$ on a lctvs is said to be *compatible with spatial scaling* if, relative to spatial magnification scaling $*$, equation (3) is satisfied for any $f \in$ Fun(V). As discussed in [8], $\{\Psi_t\}$ on Fun(V) is a T-granulometry compatible with spatial scaling if and only if it possesses a representation of the form given in equation (4). Again **G** is a generator, and **G** generates Inv[Ψ_1] by means of suprema, translations, and scalings $t*g$, $t \geq 1$, where, now, $*$ denotes spatial scaling.

Again, application requires elimination of the outer supremum. In analogy to the former cases, this means that for each g in the generator **G**, we must have $O(f, r*g) \leq O(f, t*g)$ for all $r \geq t$, which means that $t*g$ is g-open for all $t \geq 1$. The main purpose of the present paper is to characterize this relation for an important subclass of Fun(V). In doing so, we will generalize the original Matheron theorem regarding convexity to the gray-scale lctvs setting.

6. CHARACTERIZATION OF THE FUNDAMENTAL REDUCTION PROPERTY

In the context of spatial scaling, reduction of the double supremum representation for scaling-compatible T-granulometries depends on the *fundamental reduction property*: $O(t*f, f) = t*f$ for all $t \geq 1$. We characterize this property for the class of upper semicontinuous functions on the real lctvs V that possess convex, compact domains. The salient result is that such functions possess the basic reduction property if and only if they are constant on their domains. While much theory relating to mathematical morphology is algebraic in nature (viz. lattice theory), this result, as was Matheron's original theorem, is deeply topological. For instance, upper semicontinuity and compactness cannot be dropped from the hypothesis.

We approach the main result by means of a proposition, lemma, and theorem , all of which hold on topological vector spaces that are not necessarily locally convex. Only for the final theorem, the main one, do we employ local convexity. We consider an upper semicontinuous function f with compact, convex domain D. Owing to compactness, f attains its maximum m on D, and we let $M = \{x \in D: f(x) = m\}$. Upper semicontinuity assures us that M is closed.

Point b in convex set D is called an *extreme point* of D if no line segment in D contains b in its interior. Geometric intuition suggests that under magnification of a convex set D in a topological vector space there is only one way to translate D to fit within tD, $t > 1$, so that the fitting covers a given extreme point tb of tD (see Figure 1). The desired translation is determined by the global translation T that moves b to tb, namely, $T(x) = x + (t - 1)b$. This observation is formalized in the following proposition.

Proposition. For an extreme point b of a convex set D and for each $t > 1$, $D + (t - 1)b$ is the only translate of D contained in tD and containing tb. [Because D is convex, $D + (t - 1)b$ is always such a translate.]

Proof. If b is an extreme point of D, then tb is an extreme point of tD. Suppose D + z is a translate of D containing tb and D + z is a subset of tD. Write tb = z + v, v ∈ D. The point v is an extreme point of D, for if it were not it would be contained in the interior of a line L[a, b] contained in D, but this would imply that L[a, b] + z is a line in tD containing z + v = tb in its interior, which would contradict the fact that tb is an extreme point. If v = b, we are done; if not, consider the line L[b, v], which lies in D by convexity. Note that L[a, b] + z and tL[a, b] are lines in tD that contain tb. In fact, these lines lie in the 1-dimenisional closed affine subspace H of tV given by tb + s(v – b), where s ranges over the real line. We see that s = 0 yields tb ∈ H and s = t yields tv ∈ H. Finally, for s = –1,

$$tb - (v - b) = v + z - v + b = z + b \qquad (5)$$

Hence, tb lies in the interior of the line L[z + b, tv]. Since the endpoints of this line lie in tD, so does the line, which contradicts the fact that tb is an extreme point. Thus, v = b and z must equal (t – 1)b.

It is of interest to note that the converse of the proposition is true. For a proof of the converse and an alterative proof of the proposition, see [8]. The following lemma proves useful in the characterization of functions f for which t∗f is f-open.

Lemma. Suppose the following five conditions hold:

- i) D is compact and convex in V.
- ii) f is upper semicontinuous on D.
- iii) b is an extreme point of D.
- iv) w ∈ M.
- v) O(t∗f, f) = t∗f on tD for all t ≥ 1.

Then the line L[b, w], from b to w, lies in M; that is, f is maximized at every point of L[b, w].

Proof. Generally,

$$O(t*f, f) = \sup \{f_x + y: f_x + y \le t*f\} \qquad (6)$$

Since b is an extreme point, the preceding proposition shows that the opening at tb reduces to

$$O(t*f,\ f)(tb) = \sup \{f_{(t-1)b}(tb) + y:\ f_{(t-1)b} + y \le t*f\}$$

$$= \sup \{f(b) + y:\ f_{(t-1)b} + y \le t*f\} \qquad (7)$$

But, according to condition (v), O(t∗f, f)(tb) = (t∗f)(tb) = f(b), so that the last supremum is f(b). Hence, $f_{(t-1)b} \le t*f$ and

$$(t*f)(w + (t - 1)b) \ge f_{(t-1)b}(w + (t - 1)b) = f(w) = m$$

Since the range of t∗f is the same as the range of f,

$$m = (t*f)(w + (t - 1)b) = f(w/t + (t - 1)b/t) \qquad (8)$$

For t ≥1, the points w/t + (t – 1)b/t form the line L[b, w], exclusive of b, and each point is in

M. Since M is closed and b is a limit point of these points in M, $b \in M$ and the lemma is proved.

Corollary. If b is an extreme point and the hypothesized f is maximized at every point of a subset S, then f is maximized at every point of the cone C(b, S).

The next theorem extends the points at which f is maximized to the closed convex hull of the extreme points of D.

Theorem 1. Let f be an upper semicontinuous function defined on the compact, convex set D in a topological vector space V, let B be the set of extreme points for D, and suppose $O(t*f, f) = f$ for all $t \geq 1$. Then f is maximized at every point of K(B), the closed convex hull of B.

Proof. Every totally ordered (by inclusion) chain $\{K_i\}$ of convex subsets of M, the set of points at which f is maximized, possesses an upper bound U that is convex and is a subset of M. In fact, $U = \cup K_i$. By Zorn's lemma there exists a maximal convex subset S of D that is also a subset of M. Suppose there exists $b \in B - S$. According to the corollary, the convex cone C(b, S) is a subset of M. But this contradicts the maximality of S, so that B must be a subset of S. Thus, H(B), the convex hull of B, is contained in S, so that f is maximized at each point of H(B). Finally, since M is closed, $K(B) = \text{Closure}[H(B)] < \text{Closure}[S] < M$.

Up to this point the results of the present section have not required local convexity of the topological vector space V. Our final theorem does assume local convexity, for it is on a lctvs that the Krein-Milman theorem applies.

Theorem 2. Let f be an upper semicontinous function defined on a compact, convex subset D of a locally convex topological vector space V, and suppose $O(t*f, f) = t*f$ on tD for all $t \geq 1$. Then f is constant on D.

Proof. By Theorem 1, f is maximized on K(B), the closed convex hull of the extreme points of D. The Krein-Milman theorem [10] says that a convex, compact subset in a lctvs is equal to the closed convex hull of its extreme points. Hence, K(B) = D, and the theorem is proved.

We close with some remarks that are further developed in [8]:

1. In fact, under the conditions of Theorem 2, f being constant is necessary and sufficient for having $t*f$ be f-open for all $t \geq 1$. We have proven necessity; sufficiency is straigtforward.

2. We have assumed that the domain of D is convex. In [8] it is shown that, in R^n, under the weaker assumptions of upper semicontinuity and compact domain, $t*f$ is f-open for $t > 1$ if and only if f is constant and its domain is convex. The proof in [8] employs the original Matheron binary convexity theorem proven in R^n, and to this point it is an open question as to whether this stronger result holds in an abritrary lctvs.

3. In the present paper we have confined ourselves to T-granulometries, where translation-invariance is relative to both domain and range. In [8], we also consider translation invariance only in the spatial domain. Extensions of Theorem 2 apply.

7. CONCLUSION

The present paper has considered granulometries as one-parameter families of mappings on locally convex topological vector spaces and in this context has introduced a new class of gray-level granulometries compatible with domain scaling. For such granulometries there is a double-supremum representation of the kind discovered by Matheron in the binary setting. Most important, the Matheron theorem regarding convexity and binary scaling has been extended to a gray-scale theorem regarding spatial scaling and relative openness, the upshot being a characterization of those types of spatial-scaling-compatible granulometries that can be expressed as a single supremum over a family of parameterized openings by spatially scaled structuring elements. Specifically, under the assumption that the generating function primitives are upper semicontinuous and possess compact convex domains, these primitives must be flat.

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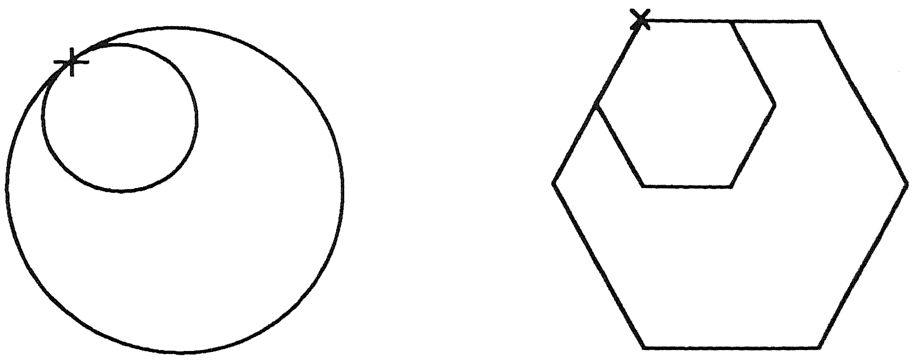

Fig. 1
Unique fitting at an extreme point for
(a) smooth convex set such as a circle
(b) a convex polygon such as a hexagon